Section 6-1, Mathematics 104

So Why do we Factor?

We're going to spend a bit more time factoring polynomials, and it is a good time to stop and reflect on why.

Let's start with an easy question

If you have two number m and n and you know that

$mn = 0$

If I tell you that $m \neq 0$ what do you now about n?

This is a subtle but important feature of real numbers.

If two numbers multiplied together equal zero, then at least one of them must be zero.

Now consider the equation

$ax^2 + bx + c = 0$

If I can factor this into linear factors, I can solve the equation by setting each linear factor equal to zero.

Example:

What is the solution to $x^2 + 3x + 2 = 0$?

Since we have $x^2 + 3x + 2 = (x+1)(x+2) = 0$

we know that if either $x+1=0$ or $x+2=0$ the original equation is solved.

This means $x=-1$ and $x=-2$ are solutions.

Let's check

$(-1)^2 + 3(-1) + 2 = 1 - 3 + 2 = 0$

$(-2)^2 + 3(-2) + 2 = 4 - 6 + 2 = 0$

This strategy works if the polynomial is of a higher degree than 2.

$$x^3 + 6x^2 + 11x + 6 = (x+1)(x+2)(x+3)$$

So the equation $x^3 + 6x^2 + 11x + 6 = 0$

has solutions $x = \{-1, -2, -3\}$

This is why we want to learn how to factor polynomials.

Because it allows us to solve polynomial equations.

Some Simple ways to factor

One of the first things to look for when trying to factor a polynomial is simple common factors.

Example:

$$8x^2 - 6x + 4$$

Each of the terms has the common factor 2, so we can immediately factor it out

Example:

$$8x^2 - 6x + 4 = 2\left(4x^2 - 3x + 2\right)$$

A common factor can also be x or x^2

$$x^3 - 2x^2 + x = x\left(x^2 - 2x + 1\right) = x\left(x-1\right)\left(x-1\right)$$

Factoring by grouping

We can take this one step further.

Any polynomial which is a common factor can be factored out

Example:

$$2x(3x-4)+(3x-4)=(2x+1)(3x-4)$$

Now sometimes by chance you get a polynomial written this way, and it is easy to see that it will factor, but usually this is not the case.

Instead we need to factor by grouping.

This is a hit or miss method. It may not work but we can always try.

If you have a polynomial with 4 terms, try grouping as follows.

$$x^3+2x^2+x+2$$

Group the first two and the last two and try to factor them individually.

$$(x^3+2x^2)+(x+2)$$

$$x^2(x+2)+(x+2)$$

Note that the second sum $x+2$ has a factor of 1

$$x^2(x+2)+1\cdot(x+2)$$

So we have a common factor $x+2$ that we can factor out

$$x^2(x+2)+1\cdot(x+2)=(x^2+1)(x+2)$$

Note: THIS WILL NOT ALWAYS WORK

Please try this one: $4x^3+4x^2-9x-9$

Solving Quadratic Equations

We know that if we have two numbers *m* and *n* for which

$$ax^2 + bx + c = (x - m)(x - n) = 0$$

That solutions for the equation are *m* and *n.*

Example:

$$2x^2 - x - 3 = (x + 1)(2x - 3)$$

So we have $x + 1 = 0 \; or \; x = -1$

and $2x - 3 = 0 \; or \; x = \frac{3}{2}$

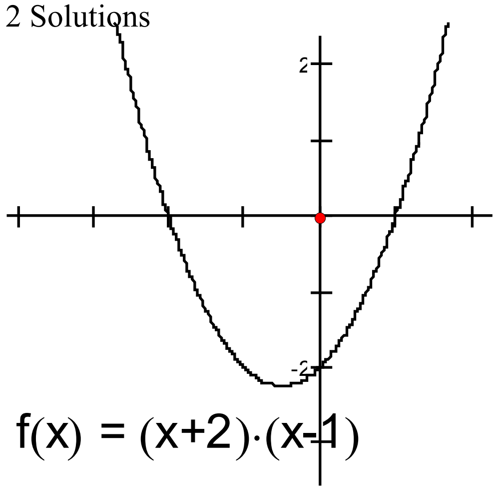
Viewing Quadratic Polynomials Graphically

We've seen before that if we graph a quadratic function like $y = x^2$ we always get a parabola.

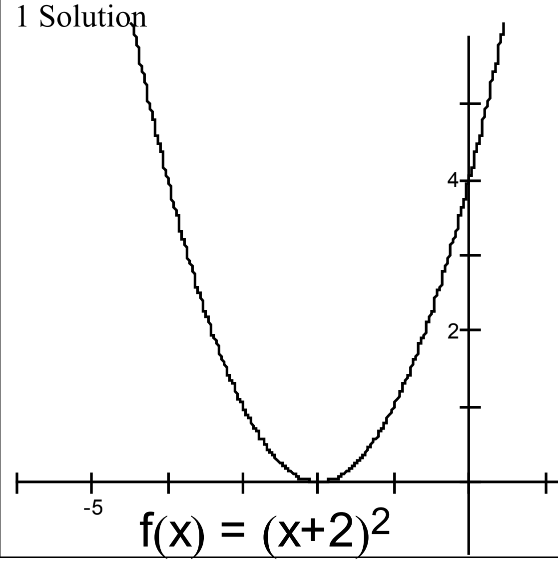
This tells us that the equation $ax^2 + bx + c = 0$ will always have either 2 solutions, 1 solution or no solution, because a solution is where the graph crosses the x axis.

Here are three examples:

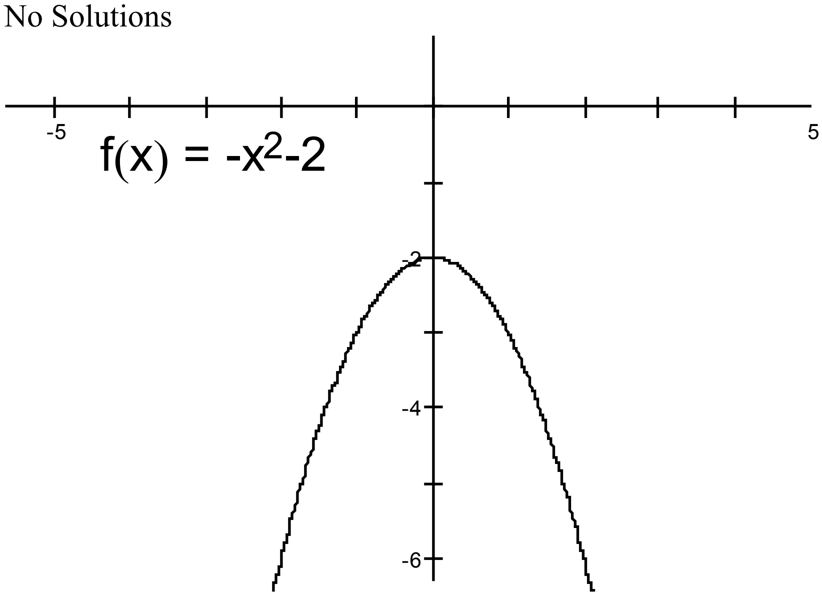
2 Solutions



1 Solution



No Solutions



Completing the square

A technique we will learn for solving these equations is called completing the square.

If we have a perfect square, example:

$$x^2 + 2x + 1 = (x+1)^2$$

Factoring is very easy.

What if have an almost perfect square

$$x^2 + 2x - 3$$

We can write this as a perfect square plus a some extra.

$$x^2 + 2x + 1 - 4$$

Now using the pattern we get

$$(x+1)^2 - 4$$

Which is in the form of the pattern difference of squares

$$(x+1)^2 - 4 = (x+1+2)(x+1-2) = (x+3)(x-1)$$

We could do this with reverse foil, but what if we had instead

$$x^2 + 2x - 2 = x^2 + 2x + 1 - 3 = (x+1)^2 - 3$$

All hope is not lost because we can write $3 = \left(\sqrt{3}\right)^2$

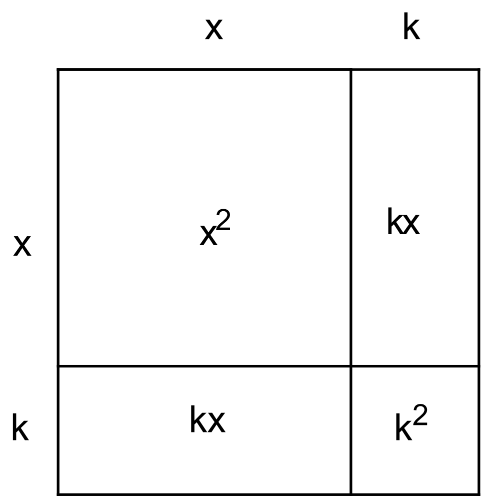
So factoring we get $x^2 + 2x - 2 = \left(x + 1 + \sqrt{3}\right)\left(x + 1 - \sqrt{3}\right)$

Please try this:

$$x^2 + 8x + 7$$

Finding the amount to complete the square.

Take a look at this visualization of completing the square:



Note that the area $A = x^2 + 2kx + k^2$

So if I know the coefficient of *x* *b*=*2k* I can find the constant term I want

$$k^2 = \frac{(2k)^2}{2} = \frac{b^2}{2}$$

This is the simple version where the coefficient of x^2, *a*=1.

If not, we have to do a little more work:

Example:

$$3x^2 + 9x - 6 = 3\left(x^2 + 3x - 2\right) = 3\left(x^2 + 3x + \frac{3^2}{2^2} - \frac{3^2}{2^2} - 2\right) =$$

$$3\left(\left(x + \frac{3}{2}\right)^2 - \left(\frac{5}{4}\right)\right) = 3\left(x + \frac{3}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{5}}{2}\right)$$

This is already getting messy.

It would be nice if would solve this once and for all in abstract manner and get a formula.

Where does the quadratic formula come from?

It comes from completing the square.   Let's see how.

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

To find the amount that completes the square we divide the coefficient of *x* by 2 and square it. We'll leave out the *a* factor for now.

$$x^2+\frac{b}{a}x+\frac{c}{a}=x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}$$

Now we have

$$\left[x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right]-\left[\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]=$$

$$\left(x+\frac{b}{2a}\right)^2-\left[\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$$

We now use the difference of squares and get

$$\left(x+\frac{b}{2a}\right)^2-\left[\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]=$$

$$\left(x+\frac{b}{2a}+\sqrt{\left(\frac{b}{2a}\right)^2+\frac{c}{a}}\right)\left(x+\frac{b}{2a}-\sqrt{\left(\frac{b}{2a}\right)^2+\frac{c}{a}}\right)$$

Re-arranging a bit we have

$$\left(x-\frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{b+\sqrt{b^2-4ac}}{2a}\right)$$

So finally we get

$$ax^2+bx+c=a\left(x-\frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{b+\sqrt{b^2-4ac}}{2a}\right)$$

So the two values we subtract from x each time are

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which we call the quadratic formula.

The main use we have for the quadratic formula is solving the equation

$$ax^2 + bx + c = 0$$

We know that the two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \; and \; x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula

When all else fails, the quadratic formula will always enable you to factor a trinomial, or discover that it is irreducible.

The quadratic formula works like this.

If $ax^2+bx+c=(x-x_1)(x-x_2)$ where we can calculate x_1 *and* x_2 using the formula

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$
$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

We write the quadratic formula this way:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Note the part of the formula under the square root sign b^2-4ac

This is called **the discriminant.**

If the discriminant is less than zero, you have the square root of a negative number.
In this case the trinomial cannot be factored.

If the discriminant is zero, x_1 *and* x_2 are the same and you have the pattern
$A^2\pm 2AB+B^2=(A\pm B)^2$

Otherwise you factor as $(x-x_1)(x-x_2)$

Example:

$x^2+x-1=?$

$$x=\frac{-1\pm\sqrt{1+4}}{2}=\frac{-1\pm\sqrt{5}}{2}$$
$$x^2+x-1=\left(x-\frac{-1+\sqrt{5}}{2}\right)\left(x-\frac{-1-\sqrt{5}}{2}\right)$$

Please try this $2x^2-3x-5$